Faculty Forum

You Cannot Conceive The Many Without The One
-Plato-

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Rock, Paper, Scissors, and Something More

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(Joint work with Robbie Bacon)

Rock-Paper-Scissors might be one of the most popular games all over the world. It can be played for many reasons. People can use it to decide who should pay the bar tab or who may have the last piece of fried chicken, or they can play it simply for fun. Its rules are easy, and it requires no equipment beyond players' hands. However, the mathematics of this game are not simple. During spring 2013, one of my students, Robbie Bacon, studied this game in our independent-study class. Afterwards, he gave a presentation on this topic at the Mathematical Association of America 2013 spring meeting at Winthrop University. This article establishes what we have learned about this game. Several variations on the game involving additional weapons also will be discussed.

In many sources the Rock-Paper-Scissors game is said to have originated in Japan. However, according to Michael E. Moore and Jennifer Sward [6], the earliest record of this game is in the Chinese book Wuzazu by Zhaozhi Xie (fl. ca. 1600), who wrote that the game dated back to the time of the Chinese Han Dynasty (206 BC – 220 AD). The game was then introduced to Japan, and spread beyond Asia by the early twentieth century when Japan increased its contact with the West (see also the work of Jepp Linhart [5]). Usually Rock-Paper-Scissors is played by two players. In it, each player shows (at the same time) his or her choice of weapons from the following three: "rock," represented by a clenched fist; "paper," represented by an open hand; and "scissors," represented by two fingers extended and separated. Each of the weapons beats one of the other two but loses to the third one. These rules form a complete cycle (A beats B, B beats C, and C beats A). Game Theory has a special term for this, a zero-sum game. If we use "1" to indicate a win, "0" for a tie, and "-1" for a loss, the sum of the points for both players in each game is always zero. For a mathematician, a zero-sum game means only one thing: the best strategy for the game is to choose one's weapons randomly (see Eric Bahel [1], Jörg Bewersdorff [2], and Len Fisher [4]). I will use some calculations to explain why.

I will begin with a brief introduction of the algebraic structure of a necessary tool, Matrix. In mathematics, a matrix is a rectangular array of numbers. These numbers are called "the entries" of a matrix. The following is an example of a 6-entry matrix:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

The subscript of each entry indicates the location of the entry. For example, a_{32} is the entry on the third row, second column. We use the number of rows and the number of columns together to indicate the dimension of a matrix, so the dimension of the above matrix is three by two (noted as 3×2), which means it contains 3 rows and 2 columns. Two matrices are said to be equal if their dimensions are the same, and the entries of each corresponding pair are equal.

Even though readers do not need to know all the operations of matrices to understand the mathematics of the game in question, I would still like to mention some basic operations to keep the algebraic structure complete. If two matrices have the same dimension, one can define the addition of these two matrices by adding the corresponding entries. Therefore, the outcome of adding two $m \times n$ matrices is still an $m \times n$ matrix. The following example shows the addition of two 2×2 matrices:

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \text{ then } A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}.$$

The subtraction of two same-dimension matrices can also be defined the same way.

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \text{ then } A - B = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \end{bmatrix}.$$

The scalar multiplication of a number and a matrix is defined by multiplying the number into each of the entries.

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \text{ then for any real number } k,\ kA = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}.$$

The multiplication of two matrices, however, is a little more complicated, but it is the main operation we will need to use to know the Rock-Paper-Scissors game. I have to start with a "one row times one column" case.

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}, \text{ then } AB = [a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31}].$$

In the above expression, a_{ij} and b_{ij} are numbers, and the small dot $(\cdot)$ is a normal multiplication of numbers. We notice that A is a 1×3 matrix and B is a 3×1 matrix, and their product AB is a 1×1 matrix. The only entry in AB is defined by the sum of all the products of the corresponding entries in A and B. This product is also known as the "dot product" in vector space. In this definition, readers can see that the product is only valid when the row has the same number of entries as the column. A general matrix may have multiple rows and multiple columns, but the product is still based on the dot product of a single row and a single column. If A is an $l \times m$

matrix and B is an $m \times n$ matrix, then the product AB is defined as the $l \times n$ matrix, in which the entry a_{ij} comes from the dot product of the i^th^ row in A and the j^th^ column in B. See the following example:

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}, \text{ then } AB = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix},$$

$$\text{in which } c_{11} = a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31}$$

(dot product of the 1st row in A and the 1st column in B),

$$c_{21} = a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31}$$

(dot product of the 2nd row in A and the 1st column in B),

$$c_{12} = a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32}$$

(dot product of the 1st row in A and the 2nd column in B), and

$$c_{22} = a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32}$$

(dot product of the 2nd row in A and the 2nd column in B).

I have to point out that in general $AB \neq BA$. Moreover, if the number of columns in B does not equal the number of rows in A, the product BA is undefined. Readers can see this from the following two examples.

If A is a 2×3 matrix and B is a 3×4 matrix, then AB is a 2×4 matrix and BA is undefined.

$$\text{If } A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \text{ then } AB = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}. \; AB \neq BA.$$

If we need to operate a product of more than two matrices, we will operate it from left to right in the order the matrices appear. For example, the product ABC shall be understood as $(AB)C$; we find the product AB first, and then multiply the new matrix by matrix C.

Now all the necessary background knowledge has been introduced, and here I'll switch my focus back to the game. However, the reader should be reminded that Matrix Theory is a lot deeper and broader than what I have demonstrated, and it is used in almost every branch of mathematics. If you are interested in digging in more, regardless of application, you may want to start with some introductory textbooks such as those by J. Costello, S. Gowdy, and A. Rash [3] or S. Warner and S. R. Costenoble [7].

Back to the Rock-Paper-Scissors game. The three weapons and the rules are well established: “rock” crushes “scissors,” “scissors” cuts “paper,” and “paper” covers “rock.” Again using “1” for a win, “0” for a tie, and “-1” for a loss, I can demonstrate all the possible combinations in the game and their “win over” situations in the following chart:

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

To read this chart, one starts with the weapon in the first column on the left. That indicates the player’s choice of weapon. One then matches it with any weapon in the first row at the top, which is the weapon used by the opponent. The number showing in the intersection of these two chosen weapons is the outcome of the match from the player’s perspective.

Opponent’s choice of weapon ↓

	Rock	**Paper**	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	**1**	0

Player’s choice of weapon → **Scissors**

The outcome of this match ↑

If we gather only the numbers, the chart forms a 3×3 matrix. We call this matrix the “payoff matrix” of this game.

$$\text{Payoff Matrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Consider the following example: assume that our opponent favors playing rock. He has a 60% chance of playing “rock,” 20% for “paper,” and 20% for “scissors.” To increase the chance of winning, we may want to play “paper” more often. Assume that our strategy is to play “rock” 25%, “paper” 50%, and “scissors” 25%. Based on these circumstances, how can we know if we actually will win more? We can use the multiplication of matrices. However, to be sure that the multiplication makes sense, we have to write the percentages of our strategy as a row matrix, and those of our opponent's strategy as a column matrix:

Our strategy= $[0.25 \quad 0.5 \quad 0.25]$; the opponent's strategy= $\begin{bmatrix} 0.6 \\ 0.2 \\ 0.2 \end{bmatrix}$

Multiplying our strategy from the left with the payoff matrix and with the opponent's strategy from the right, we will finally end up with a 1×1 matrix. The only entry of that matrix is called the expected value:

$$[0.25 \quad 0.5 \quad 0.25]\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0.6 \\ 0.2 \\ 0.2 \end{bmatrix} = [0.1]$$

I want to point out that the expected value is not a probability. Instead, it is a weighted average of "our winnings" and "our losses." Readers can think of it this way. We get 1 point if we win, and we get (-1) point if we lose. The expected value is the theoretical average of all points we received after playing many games. If the number is zero, our strategy does not provide us any advantage. In other words, we and our opponent share the same weight of winning or losing. If the number is positive, no matter how big or small, we will have more wins than losses; hence, our strategy is in favor of our winning. If the number is negative, our strategy actually makes us lose more than we win. In the above example, the expected value is positive, so that means this strategy does make us win more. For more discussion about expected value see Costello, Gowdy, and Rash [3] and Warner and Costenoble [7].

Let me use the above example to analyze something more. Assume that the opponent still uses the same strategy. Can we increase our expected value by improving our strategy? Yes, we can. Actually, we would like to find the strategy that will provide us the biggest expected value. In Game Theory, this is called "solving a game," and it is the main goal for every player in every game. To solve this game based on the given information, we can approach it just like solving other algebra problems: we will use variables to help us. If we use x for the probability of playing "rock," y for "paper," and z for "scissors," we can write our strategy as a row matrix: $[x \quad y \quad z]$. Remember, all these variables indicate a probability, so they have to be bounded between $0\%(= 0)$ and $100\%(= 1)$. Replacing our strategy matrix with the new algebraic row matrix in the previous matrix multiplication, we have:

$$[x \quad y \quad z]\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0.6 \\ 0.2 \\ 0.2 \end{bmatrix} = [0x + 0.4y - 0.4z] = [0.4y - 0.4z].$$

Apparently, the percentage of playing "rock" will not affect the expected value. However, a bigger y results in a bigger expected value while a smaller z leads to the same result. Therefore, the best strategy for us, based on the information, is to play "paper" all the time: $[0 \quad 1 \quad 0]$.

Theoretically, we do not have any advantage if our opponent chooses the weapon at random (which means each weapon has the same probability to be played). Check the following calculation:

$$[x \quad y \quad z]\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = [0].$$

Readers can see that no matter what our strategy is, our expected value remains zero. However, in real life it is almost impossible to play the game randomly. Some players consistently start with a specific weapon, and some players tend to play one weapon more than the other two. And that is what we can take advantage of. (I am not saying one should—but if one wants to, one can start with tracing the opponent's game history or game habit! See my notes at the end of this article.)

A game is called a fair game if the expected value is zero when all players play by random choices. That means the game itself is not in favor of any of the players. And a game is called a balanced game if the sum of all entries in each row and each column in the payoff matrix is zero. The Rock-Paper-Scissors game is not only a fair game, but also a balanced one. Readers can also see it this way: the weapon "rock" beats "scissors" but loses to "paper"; "paper" beats "rock" but loses to "scissors"; and "scissors" beats "paper" but loses to "rock." Obviously each weapon has an equal ability to beat other weapons. That's another way to tell whether a game is balanced.

What if one more weapon is added into this game? This new game will not be balanced. Let us choose one weapon, and there will be three other weapons left. Three is an odd number and cannot be separated evenly. In other words, in a four-weapon game every weapon will not be able to beat and lose to other weapons evenly; it either beats two other weapons and loses to only one, or beats only one and loses to two. As a matter of fact, any even-number-weapon game is not a balanced game. Only an odd-number-weapon game can be a balanced game if the rules are well designed. Using the same analysis mentioned above one easily can see this fact (see the works by Bahel [1] and Bewersdorff [2]).

Now let me introduce an actual four-weapon game. In France, there is a popular game called Pierre-Papier-Ciseaux-Puits (stone, paper, scissors, well). The hand gestures of the first three weapons are the same as the regular Rock-Paper-Scissors game. For "well," the gesture is made by forming a circle with the thumb and index finger to represent the opening of the well. The rules for this game are: "stone" crushes "scissors," "scissors" cuts "paper," "paper" covers both "stone" and "well," and both "stone" and "scissors" fall into "well." We already know that this game cannot be balanced. To see this from a mathematical point of view, one can check the sum of all the numbers in a row (or a column) in the payoff matrix of this game. To make it easier to read, we will first create a chart to indicate both the weapons and the outcomes.

	Stone	Paper	Scissors	Well
Stone	0	-1	1	-1
Paper	1	0	-1	1
Scissors	-1	1	0	-1
Well	1	-1	1	0

Remember that the payoff matrix is composed of all the numbers in this chart. And the sum of all the numbers in a row or a column is either "1" or "-1." This may lead one to ask whether the game possibly can be fair. Let us check the expected value.

$$[.25 \quad .25 \quad .25 \quad .25]\begin{bmatrix} 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} .25 \\ .25 \\ .25 \\ .25 \end{bmatrix} = [0].$$

From the above calculation we notice that the expected value is zero if the game is played by random choices. In other words, it is a fair game!

Now I will check to see if playing randomly is the best strategy. In this case let w, x, y, and z be the probability for us to play "stone," "paper," "scissors," and "well," respectively. The following calculation gives us an algebraic expression of the expected value:

$$[w \quad x \quad y \quad z]\begin{bmatrix} 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} .25 \\ .25 \\ .25 \\ .25 \end{bmatrix} = \left[-\frac{w}{4} + \frac{x}{4} - \frac{y}{4} + \frac{z}{4}\right].$$

From the algebraic expression we can see that if we want the expected value to be positive, and as big as possible, we would play only "paper" or "well." It can be only one of them, or any mixed strategy of both weapons. However, we have to assume that our opponent also knows about this, and would play "paper" or "well" too. Therefore, we may want to adjust our strategy a little bit. Let's assume that our opponent will play "paper" 50% and "well" 50%. Based on this assumption, we calculate the expected value again:

$$[w \quad x \quad y \quad z]\begin{bmatrix} 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ .50 \\ 0 \\ .50 \end{bmatrix} = \left[-w + \frac{x}{2} - \frac{z}{2}\right].$$

This time, the algebraic expression suggests that we should play “paper” every time.

The mathematical analysis above only gives us a rough idea around which we can plan our strategy. However, when actually playing a game not many people would play only one weapon. If you do so, your opponent quickly will figure it out and form another strategy to beat you, or at least force you to change your choice of weapons. Hence, it might be better to play other weapons occasionally to make your strategy a little unpredictable. It is fair to say that this game is more psychological than mathematical (see Bewersdorff [2] and Fisher [4]).

There is another four-weapon game I learned at the 2013 Joint Mathematics Meeting in Boston: Rock-Paper-Scissors-Bomb. It was introduced in a math-education session. To play the weapon “bomb,” one stretches one’s thumb with the other four fingers bending inward, forming a shape of a bomb with a fuse at the top. The rules for this game are: “rock” crashes “scissors”; “scissors” cuts “paper” and the fuse of the “bomb”; “paper” covers “rock”; and “bomb” smashes both “paper” and “rock.” This time the seemingly more advantageous weapons are “scissors” and “bomb.” However, several presenters in that session mentioned that when they introduced this game into their classrooms, they found the most winning weapon to be “scissors.” They reasoned that to most of the students “bomb” is a new weapon, and to the students it seems to have the advantage, so at the beginning they tend to play “bomb” a lot. But very soon, a group of students realize this tendency among their classmates and start to play “scissors” instead. So in almost every case “scissors” is the big winner at the end of the class. This serves as another example of how psychology trumps mathematical theory.

Is there a five-weapon game? Sure! The most famous five-weapon game is probably Rock-Paper-Scissors-Lizard-Spock, mentioned in the popular TV show “The Big Bang Theory.” This game was actually invented by Sam Kass and Karen Bryla [8]. The two additional weapons are played by forming the hand into a sock-puppet-like mouth, representing the lizard’s head for “lizard,” and by forming a Star Trek Vulcan salute for “Spock.” In a regular Rock-Paper-Scissors game, the probability of ending with a tie is one-third, or approximately 33%. And according to Kass [8], if you play with someone you know well enough, the probability will increase to 75-80%. He invented Rock-Paper-Scissors-Lizard-Spock to reduce that probability. Readers already learned earlier that an odd-number-weapon game is balanced, so its rules form a complete cycle: “scissors” cuts “paper”; “paper” covers “rock”; “rock” crashes “lizard”; “lizard” poisons “Spock”; “Spock” smashes “scissors”; “scissors” decapitates “lizard”; “lizard” eats “paper”; “paper” disproves “Spock”; “Spock” vaporizes “rock”; and, as it always has, “rock” crashes “scissors.” During our independent study, Mr. Bacon and I noticed that not everyone can play “Spock” easily because it requires an unnatural hand gesture. So if you really want to play this game, you may want to practice “Spock” at home first.

Another commonly played five-weapon game is Rock-Paper-Scissors-Well-Bull, which is well known in Germany. The weapon “bull” is played by forming a fist and stretching out the thumb and the little finger, forming the horns of a bull. The additional rules go like this: “bull” eats “paper” and drinks from “well,” but is smashed by “rock” and stabbed by “scissors.”

Is there any game with more than five weapons? Yes, indeed! And more than one! David C. Lovelace [9] has created variations on the Rock-Paper-Scissors game with 7 weapons, 9 weapons, 11 weapons, 15 weapon, 25 weapons, and 101 weapons! As the number of weapons increases, some may require a full body move to play. Also, with rules too complicated to remember, even Lovelace himself writes on his website, “I highly doubt anyone will actually even want to attempt to play it.” Interested readers may check his website for his variations on the Rock-Paper-Scissors game.

Note

Even though we all know how to play the Rock-Paper-Scissors game, we may not play it 100% correctly, at least according to the World Rock-Paper-Scissors Society [10]. On its website, the society establishes the official rules, the official hand gestures, and the official process (including the “prime,” the ritual used to get players in sync with each other so they can deliver their throws simultaneously; the “approach,” the transition phase between the final prime and the final stage when players deliver their throws; and the final stage, the “delivery”). These instructions are followed in the World RPS Tournament. But there is another reason you may want to check out the society’s website. Remember I mentioned the psychological effect? The World Rock-Paper-Scissors Society uses statistical data gathered in all the tournaments to back up this hypothesis. Working from their analysis of metadata, the society offers an essay titled “How to Beat Anyone at Rock-Paper-Scissors” to suggest different techniques in different conditions. I recommend you have a look, and have fun!

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